The Jumping Phenomenon of Hodge Numbers

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Abstract

Let X be a compact complex manifold, consider a small deformation $\phi : \mathcal{X} \to B$ of X, the dimension of the Dolbeault cohomology groups $H^q(X_t, \Omega^p_{X_t})$ may vary under this defromation. This paper will study such phenomenons by studying the obstructions to deform a class in $H^q(X, \Omega^p_X)$ with the parameter t and get the formula for the obstructions.

1 Introduction

Let X be a compact complex manifold and $\phi : \mathcal{X} \to B$ be a family of complex manifolds such that $\phi^{-1}(0) = X$. Let $X_t = \phi^{-1}(t)$ denote the fibre of ϕ above the point $t \in B$. We denote by $\mathcal{O}_X$ and Ω^p_X the sheaves of germs of X of holomorphic functions and p-forms respectively. Recall $h^{p,q} = dim_{\mathbb{C}} H^q(X, \Omega^p_X)$ and $P_m = dim H^0(X, (\Omega^n_X)^{\bigotimes m})$ where $n = dim_{\mathbb{C}} X$. S.Iitaka proposed a problem whether all P_m are deformation invariants [1]. This problem was solved by Iku Nakamura in his paper [2], and actually he gave us some examples of small deformations of complex parallelisable manifold (by a complex parallelisable manifold we mean a compact complex manifold with the trivial holomorphic tangent bundle) such that the hodge numbers of the fibre of the family jump in these deformations.

In this paper, we will study such phenomenons from the viewpoint of obstruction theory. More precisely, for a certain small deformation $\mathcal{X}$ of X parametrized by a basis B and a certain class $[\alpha]$ of the Dolbeaut cohomology group $H^q(X, \Omega^p_X)$, we will try to find out the obstruction to extending it to an element of the relative Dolbeaut cohomology group $H^q(\mathcal{X}, \Omega^p_{\mathcal{X}/B})$. We will call those elements which have non trivial obstruction the obstructed elements.

In §2 we will summarize the results of Grauert's Direct Image Theorems and we will try to explain why we need to consider the obstructed elements. Actually, we will see that these elements will play an important role when we study the jumping phenomenon of Hodge numbers. Because we will see that the existence of the obstructed elements is a necessary and sufficient condition for the variation of the Hodge diamond.

In §3 we will get a formula for the obstruction to the extension we mentioned above.

Theorem 3.3 *Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the nth order deformation of X. For arbitrary $[\alpha]$ belongs to $H^q(X, \Omega^p)$, suppose we can extend $[\alpha]$ to order $n-1$ in $H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}})$. Denote such element by $[\alpha_{n-1}]$. The obstruction of the extension of $[\alpha]$ to nth order is given by:*

$$o_{n,n-1}(\alpha) = d_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner (\alpha_{n-1}) + \kappa_n \llcorner \circ d_{X_{n-1}/B_{n-1}}(\alpha_{n-1}),$$

where κ_n is the nth order Kodaira-Spencer class and $d_{X_{n-1}/B_{n-1}}$ is the relative differential operator of the $n-1$th order deformation.

In §4 we will use this formula to study carefully the example given by Iku Nakamura, i.e. the small deformation of the Iwasama manifold and discuss some phenomenons.

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2 Grauert's Direct Image Theorems and Deformation theory

In this section, let us first review some general results of deformation theory. Let X be a compact complex manifold. The manifold X has an underlying differential structure, but given this fixed underlying structure there may be many different complex structures on X. In particular, there might be a range of complex structures on X varying in an analytic manner. This is the object that we will study.

Definition 1.0 A deformation of X consists of a smooth proper morphism $\phi : \mathcal{X} \to B$, where $\mathcal{X}$ and B are connected complex spaces, and an isomorphism $X \cong \phi^{-1}(0)$, where $0 \in B$ is a distinguished point. We call $\mathcal{X} \to B$ a family of complex manifolds.

Although B is not necessarily a manifold, and can be singular, reducible, or non-reduced, (e.g. $B = Spec\, \mathbb{C}[\varepsilon]/(\varepsilon^2)$), since the problem we are going to research is the phenomenon of the jumping of the Dolbeaut cohomology, we may assume that $\mathcal{X}$ and B are complex manifolds.

In order to study the jumping of the Dolbeaut cohomology, we need the following important theorem (one of the Grauert's Direct Image Theorems).

Theorem 1.1 *Let X, Y be complex spaces, $\pi : X \to Y$ a proper holomorphic map. Suppose that Y is Stein, and let $\mathcal{F}$ be a coherent analytic sheaf on X. Let Y_0 be a relatively compact open set in Y. Then, there is an integer $N > 0$ such that the following hold.*
I. There exists a complex

$$\mathcal{E}^{\cdot} : ... \to \mathcal{E}^{-1} \to \mathcal{E}^{0} \to ... \to \mathcal{E}^{N} \to 0$$

of finitely generated locally free $\mathcal{O}_{Y_0}$-modules on Y_0 such that for any Stein open set $W \subset Y_0$, we have

$$H^q(\Gamma(W, \mathcal{E}^{\cdot})) \simeq \Gamma(W, R^q\pi_*(\mathcal{F})) \simeq H^q(\pi^{-1}(W), \mathcal{F}) \qquad \forall q \in \mathbb{Z}.$$

II. (Base Change Theorem). Assume, in addition, that $\mathcal{F}$ is π-flat [i.e. $\forall x \in X$, the stalk $\mathcal{F}_x$ is flat over as a module over $\mathcal{O}_{Y,\pi(x)}$]. Then, there exists a complex

$$\mathcal{E}^{\cdot} : 0 \to \mathcal{E}^{0} \to \mathcal{E}^{1} \to ... \to \mathcal{E}^{N} \to 0$$

of finitely generated locally free $\mathcal{O}_{Y_0}$-sheaves $\mathcal{E}^p$ with the following property:

Let S be a Stein space and $f : S \to Y$ a holomorphic map. Let $X^{'} = X \times_Y S$ and $f^{'} : X^{'} \to X$ and $\pi^{'} : X^{'} \to S$ be the two projections. Then, if T is an open Stein subset of $f^{-1}(Y_0)$, we have, for all $q \in \mathbb{Z}$

$$H^q(\Gamma(T, f^*(\mathcal{E}^{\cdot}))) \simeq \Gamma(T, R^q\pi^{'}_*(\mathcal{F}^{'})) \simeq H^q(\pi^{'-1}(T), \mathcal{F}^{'})$$

where $\mathcal{F}^{'} = (f^{'})^*(\mathcal{F})$.

Let X,Y be complex spaces, $\pi : X \to Y$ a proper map. Let $\mathcal{F}$ be a π-flat coherent sheaf on X. For $y \in Y$, denote by $\mathcal{M}_y$ the $\mathcal{O}_Y$-sheaf of germs of holomorphic functions "vanishing at y": the stalk of $\mathcal{M}_y$ at y is the maximal ideal of $\mathcal{O}_{Y,y}$; that at $t \neq y$" is $\mathcal{O}_{Y,t}$. We set $\mathcal{F}(y)$ = analytic restriction of $\mathcal{F}$ to $\pi^{-1}(y) = \mathcal{F} \bigotimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathcal{M}_y)$. Since we just need to study the local properties, we may assume, in view of Theorem 1.0, part II, that there is a complex

$$\mathcal{E}^{\cdot} :\ 0 \longrightarrow \mathcal{O}_Y^{P_0} \xrightarrow{d^0} \mathcal{O}_Y^{P_1} \xrightarrow{d^1} \cdots \xrightarrow{d^{N-1}} \mathcal{O}_Y^{P_N} \xrightarrow{d^N} 0$$

with the base change property in Theorem 1.0, part II. In particular, if $y \in Y$, we have

$$H^q(\pi^{-1}(y), \mathcal{F}(y)) \simeq H^q(\mathcal{E}^{\cdot} \otimes (\mathcal{O}_Y/\mathcal{M}_y)).$$

Apply what we discussed above to our case $\phi : \mathcal{X} \to B$, we get the following. There is a complex of vector bundles on the basis B, whose cohomology groups at the point identifies to the cohomology groups of the fiber X_b with values in the considered vector bundle on $\mathcal{X}$, restricted to X_b. Therefore, for arbitrary p, there exists a complex of vector bundles $(E^{\cdot}, d^{\cdot})$, such that for arbitrary $t \in B$, $H^q(X_t, \Omega^p_{X_t}) = H^q(E^{\cdot}_t) = Ker(d^q)/Im(d^{q-1})$.

Via a local trivialisation of the bundle E^i, the differential of the complex $E^{\cdot}$ are represented by matrices with holomorphic coefficients, and follows from the lower semicontinuity of the rank of a matrix with variable coefficients , it is easy to check that the function $dim_{\mathbb{C}}Ker(d^q)$ and $-dim_{\mathbb{C}}Im(d^q)$ are upper semicontinuous on B. Therefore the function $dim_{\mathbb{C}}H^q(E^{\cdot}_t)$ is also upper semicontinuous. It seems that either the increasing of $dim_{\mathbb{C}}Im(d^{q-1})$ or the

decreasing of $dim_{\mathbb{C}}Ker(d^q)$ will cause the jumping of $dim_{\mathbb{C}}H^q(E_t^{\cdot})$, however, because of the following exact sequence:

$$0 \to Ker(d^q)_t \to E_t^q \to Im(d^q)_t \to 0 \qquad \forall t,$$

which means the variation of $-dim_{\mathbb{C}}Im(d^q)$ is exactly the variation of $dim_{\mathbb{C}}Ker(d^q)$, we just need to consider the variation of $dim_{\mathbb{C}}Ker(d^q)$ for all q.

In order to study the variation of $dim_{\mathbb{C}}Ker(d^q)$, we need to consider the following problem. Let α be an element of $Ker(d^q)$ at $t=0$, we try to find out the obstruction to extending it to an element which belongs to $Ker(d^q)$ in a neighborhood of 0. Such kind of extending can be studied order by order. Let $\mathcal{E}_0^q$ be the stalk of the assocaited sheaf of E^q at 0. Let m_0 be the maximal idea of $\mathcal{O}_{B,0}$. For arbitrary positive intergal n, since d^q can be represented by matrices with holomorphic coefficients, it is not difficult to check $d^q(\mathcal{E}_0^q \otimes_{\mathcal{O}_{B,0}} m_0^n) \subset \mathcal{E}_0^{q+1} \otimes_{\mathcal{O}_{B,0}} m_0^n$. Therefore the complex of the vector bundles $(E^{\cdot}, d^{\cdot})$ induces the following complex:

$$0 \to \mathcal{E}_0^0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n \xrightarrow{d^0} \mathcal{E}_0^1 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n \xrightarrow{d^1} \dots \xrightarrow{d^{N-1}} \mathcal{E}_0^N \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n \xrightarrow{d^N} 0.$$

Definition 2.2 Those elements of $H^{\cdot}(E_0^{\cdot})$ which can not be extended are called *the first class obstructed elements*.

Next, we will show the obstructions of the extending we mentioned above. For simplicity, my may assume that $dim_{\mathbb{C}}B = 1$, suppose α can be extended to an element α_{n-1} such that $j_0^{n-1}(d^q(\alpha_{n-1}))(t) = 0$, then α_{n-1} can be considered as the $n-1$ order extension of α. Here $j_0^{n-1}(d^q(\alpha_{n-1}))(t)$ is the $n-1$ jet of $d^q(\alpha_{n-1})$ at 0.

Define a map $o_n^q : H^q(\mathcal{E}_0^{\cdot} \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n) \to H^{q+1}(E_0^{\cdot})$ by

$$[\alpha_{n-1}] \longmapsto [j_0^n(d^q(\alpha_{n-1}))(t)/t^n].$$

At first, we need to check o_n^q is well defined. So we need to show that $[j_0^n(d^q(\alpha_{n-1}))(t)/t^n]$ is d^{q+1}-closed. Via a local trivialization of the bundles E^i, the differentials of the complex $E^{\cdot}$ are represented by matrices with holomorphic coefficients, and from the lower semi-continuity of the rank of a matrix with variable coefficients, we may assume that there always exists $(\sigma_1^{q+1}, ..., \sigma_l^{q+1})$ which are sections of E^{q+1} such that $(\sigma_1^{q+1}|_{t=0}, ..., \sigma_l^{q+1}|_{t=0})$ form a basis of $Ker(d^{q+1} : E_0^{q+1} \to E_0^{q+2})$ and $Ker(d^{q+1} : E^{q+1} \to E^{q+2}) \subset Span\{\sigma_j^{q+1}\}$. So we can write $d^q(\alpha_{n-1}) = \sum_j f_j \sigma_j^{q+1}$.

Since $j_0^{n-1}(d^q(\alpha_{n-1}))(t)$=0, we have $f_j = 0$ and $\frac{\partial^i f}{\partial t^i} = 0$, $i = 1..n-1$.

$$\frac{\partial^n}{\partial t^n}(d^q(\alpha_{n-1}))|_{t=0} = \sum_j \frac{\partial^n f_j}{\partial t^n}\sigma_j^{q+1}|_{t=0}+...+\sum_j f_j\frac{\partial^n}{\partial t^n}(\sigma_j^{q+1})|_{t=0} = \sum_j \frac{\partial^n f_j}{\partial t^n}\sigma_j^{q+1}|_{t=0},$$

therefore

$$d^{q+1}(\frac{\partial^n}{\partial t^n}(d^q(\alpha))|_{t=0}) = d^{q+1}(\sum_j \frac{\partial^n f_j}{\partial t^n}\sigma_j^{q+1}|_{t=0}) = 0,$$

which means $\frac{\partial^n}{\partial t^n}(d^q(\alpha_{n-1}))|_{t=0}$ is d^{q+1}-closed.

Next we are going to show that the equivalent class of $\frac{\partial^n}{\partial t^n}(d^q(\alpha_{n-1}))|_{t=0}$ in $H^{q+1}(E_0^\cdot)$ depends only on $j_0^{n-1}(\alpha_{n-1})(t)$. Let $(\sigma_1^q, ..., \sigma_k^q)$ be a bases of E^q, we only need to show that if $j_0^{n-1}(\alpha_{n-1})(t) = 0$, then $\frac{\partial^n}{\partial t^n}(d^q(\alpha_{n-1}))|_{t=0}$ belongs to $Im(d^q : E^q \to E^{q+1})$. Indeed, we can write $\alpha_{n-1} = \sum_j f_j\sigma_j^q$ while $f_j(0) = 0$, $\frac{\partial^i f}{\partial t^i} = 0$, $i = 1...n-1$, then,

$$\frac{\partial^n}{\partial t^n}(d^q(\alpha_{n-1})) = \frac{\partial^n}{\partial t^n}(\sum_i f_i d^q(\sigma_i^q)) = \sum_i \frac{\partial^n f_i}{\partial t^n}d^q(\sigma_i^q)+...+\sum_i f_i\frac{\partial^n}{\partial t^n}(d^q(\sigma_i^q)).$$

Therefore, $\frac{\partial^n}{\partial t^n}(d^q(\Omega))|_{t=0} = \sum_i \frac{\partial^n f_i}{\partial t^n}d^q(\sigma_i^q)|_{t=0}$, which belongs to $Im(d^q : E^q \to E^{q+1})$.

At last, we are going to show that the equivalent class of $\frac{\partial^n}{\partial t^n}(d^q(\Omega))|_{t=0}$ in $H^{q+1}(E_0^\cdot)$ depends only on the equivalent class of α_{n-1} in $H^q(\mathcal{E}_0^\cdot \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n)$. Actually, we only need to show that if α_{n-1} belongs to $Im(d^{q-1} : \mathcal{E}_0^{q-1} \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n \to \mathcal{E}_0^q \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n)$, we will have $\frac{\partial^n}{\partial t^n}(d^q(\alpha_{n-1}))|_{t=0}$ belongs to $Im(d^q : E^q \to E^{q+1})$. In fact, let $\alpha'_{n-1} = d^{q-1}(\sum_j f_j\sigma_j^{q-1})$ such that $j_0^{n-1}(\alpha'_{n-1})(t) = j_0^{n-1}(\alpha_{n-1})(t)$. From the discussion above, we have

$$\frac{\partial^n}{\partial t^n}(d^q(\alpha_{n-1}))|_{t=0} = \frac{\partial^n}{\partial t^n}(d^q(\alpha'_{n-1}))|_{t=0} = \frac{\partial^n}{\partial t^n}(d^q(d^{q-1}(\sum_j f_j\sigma_j^{q-1}))) = 0$$

in $H^q(E^\cdot)$.

Remark It seems that $j_0^n(d^q(\alpha_{n-1}))(t)/t^n$ depends on the connection of E^{q+1}. But, by using an induction argument, it is not difficult to prove that if $j_0^i(d^q(\alpha_{n-1}))(t) = 0, \forall i < n$, then $j_0^n(d^q(\alpha_{n-1}))(t)$ is independent of the choice of the connection of E^{q+1}.

There is natural a map $\rho_i^q : H^q(E_0^\cdot) \to H^q(\mathcal{E}_0^\cdot \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^{i+1})$ given by

$$[\sigma] \longmapsto [t^i\sigma], \forall[\sigma] \in H^q(E_0^\cdot).$$

Denote the map $\rho_i^{q+1}\circ o_n^q : H^q(\mathcal{E}_0^{\cdot}\otimes_{\mathcal{O}_{B,0}}\mathcal{O}_{B,0}/m_0^n) \to H^{q+1}(\mathcal{E}_0^{\cdot}\otimes_{\mathcal{O}_{B,0}}\mathcal{O}_{B,0}/m_0^{i+1}), \forall i \leq n$ by $o_{n,i}^q$.

Next we will show that, for arbitrary i, $0 < i \leq n$, α_{n-1} can be extended to α_n which is the nth order extension of α such that $j_0^{i-1}(\alpha_n - \alpha_{n-1})(t) = 0$ if and only if $o_{n,n-i}^q([\alpha_{n-1}])$ is trivial. For necessarity, $(\alpha_n - \alpha_{n-1})(t)/t^i$ is supposed to be the preimage of $o_{n,n-i}^q([\alpha_{n-1}])$, so $o_{n,n-i}^q([\alpha_{n-1}])$ is trivial. Therefore we just need to check whether it is sufficient. In fact, if $o_{n,n-i}^q([\alpha_{n-1}])$ is trivial, then there exists a section β of $\mathcal{E}_0^{\cdot} \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^{i+1}$ such that $d^q(\beta) = o_{n,n-i}^q([\alpha_{n-1}])$. Then it is not difficult to check that $\alpha_{n-1} - t^i\tilde{\beta}$ is an n th order extension of α that we need, where $\tilde{\beta}$ is an extension of β in the neighborhood of 0. Therefore we have the following proposition.

Proposition 2.3 *Let α_{n-1} be an $n-1$ th order extension of α, for arbitrary i, $0 < i \leq n$, α_{n-1} can be extended to α_n which is the nth order extension of α such that $j_0^{i-1}(\alpha_n - \alpha_{n-1})(t) = 0$ if and only if $o_{n,n-i}^q([\alpha_{n-1}]) = 0$.*

In the following, we will show that the obstructions $o_n^q([\alpha_{n-1}])$ also play an important role when we consider about the jumping of $dim_{\mathbb{C}}Im(d^q)$. Note that $dim_{\mathbb{C}}Im(d^q)$ jumps if and only if there exist a section β of $dim_{\mathbb{C}}Ker(d^{q+1})$, such that β_0 is not exact while β_t is exact for $t \neq 0$.

Definition 2.4 Those nontrivial elements of $H^{\cdot}(E_0^{\cdot})$ that can always be extended to a section which is only exact at $t \neq 0$ are called *the second class obstructed elements*.

Note that if α is exact at $t = 0$, it can be extended to an element which is exact at every point. So the definition above does not depend on the element of a fixed equivalent class.

Proposition 2.5 *Let $[\beta]$ be an nontrivial element of $H^{q+1}(E_0^{\cdot})$. Then $[\beta]$ is a second class obstructed element if and only if there exist $n \geq 0$ and α_{n-1} in $H^q(\mathcal{E}_0^{\cdot} \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n)$ such that $o_n^q([\alpha_{n-1}]) = [\beta]$.*

Proof. If $o_n^q([\alpha_{n-1}]) = [\beta]$, then $j_0^n(d^q(\alpha_{n-1}))(t)/t^n$ is the extension we need. On the contrary, if $[\beta]$ is a second class obstructed element. There exist $\tilde{\beta}$ such that $\tilde{\beta}_t$, $t \neq 0$ is exact. Then $(d^q)^{-1}(\tilde{\beta})$ is a meromorphic section which has a pole at $t = 0$. Let n be the degree of $(d^q)^{-1}(\tilde{\beta})$. Then let $\alpha_{n-1} = t^n(d^q)^{-1}(\tilde{\beta})$. It is easy to check that $o_n^q([\alpha_{n-1}]) = [\beta]$. $\square$

Proposition 2.6 *Let α_{n-1} be an element of $H^q(\mathcal{E}_0^{\cdot} \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^n)$ such*

that $o^q_n([\alpha_{n-1}]) \neq 0$. Then there exists $n^{'} \leq n$ and $\alpha^{'}$ be an element of $H^q(\mathcal{E}^{\cdot}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m_0^{n^{'}})$, such that $\rho^{q+1}_{n^{'}-1} \circ o^q_n([\alpha_{n-1}]) = o^q_{n^{'},n^{'}-1}([\alpha^{'}]) \neq 0$.

Proof. If $o^q_{n,n-1}([\alpha_{n-1}]) \neq 0$, then $n^{'} = n$ and $\alpha^{'} = \alpha_{n-1}$. Otherwise, there exists $\alpha^{'}_1$, such that $d^q(\alpha^{'}_1) = \rho^{q+1}_{n-1} \circ o^q_n([\alpha_{n-1}])$. Note that $o^q_{n-1,n-2}([\alpha^{'}_1]) = \rho^{q+1}_{n-2} \circ o^q_n([\alpha_{n-1}]) = o^q_{n,n-2}([\alpha_{n-1}])$. If we go on step by step as above, we can always get the $n^{'}$ and $\alpha^{'}$ for there is at least one of the $o^q_{n,i}([\alpha_{n-1}])$ is nontrivial. □

This proposition tells us that althought $o^q_n([\alpha_{n-1}]) \neq 0$ does not mean that $o^q_{n,n-1}([\alpha_{n-1}]) \neq 0$, we can always find $\alpha^{'}$ such that $o^q_n([\alpha_{n-1}])$ comes from obstuctions like $o^q_{n,n-1}([\alpha^{'}])$. Therefore we can get the following corollary immediately from Proposition 2.5 and Proposition 2.6.

Corollary 2.7 *Let $[\beta]$ be an nontrivial element of $H^{q+1}(E^{\cdot}_0)$. Then $[\beta]$ is a second class obstructed element if and only if there exist $n \geq 0$ and α_{n-1} in $H^q(\mathcal{E}^{\cdot}_0 \otimes_{\mathcal{O}_{B,0}} \mathcal{O}_{B,0}/m^n_0)$ such that $o^q_{n,n-1}([\alpha_{n-1}]) = \rho^{q+1}_{n-1}([\beta])$.*

Let us come back to our problem, suppose α can be extended to an element α_{n-1} such that $j^{n-1}_0(d^q(\alpha_{n-1}))(t) = 0$, since what we care is whether α can be extended to an element which belongs to $Ker(d^q)$ in a neighborhood of 0. So, if we have an nth order extension α_n of α, it is not necessary that $j^{i-1}_0(\alpha_n - \alpha_{n-1})(t) = 0, \forall i, 1 < i < n$. What we need is just $j^0_0(\alpha_n - \alpha_{n-1})(t) = 0$ which means α_n is an extension of α. So the "real" obstructions come from $o^q_{n,n-1}([\alpha_{n-1}])$. Since these obstructions is so important when we consider the problem of variation of hodge numbers, we will try to find out an explicit calculation for such obstructions in next section.

3 The Formula for the Obstructions

We are going to prove in this section an explicit formula (Theorem 3.3) for the abstract obstructions described above. Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. For every integer $n \geq 0$, denote by $B_n = Spec\, \mathcal{O}_{B,0}/m^{n+1}_0$ the nth order infinitesimal neighborhood of the closed point $0 \in B$ of the base B. Let $X_n \subset \mathcal{X}$ be the complex space over B_n. Let $\pi_n : X_n \to B_n$ be the nth order deformation of X. In order to study the jumping phenomenon of Dolbeaut cohomology groups, for arbitrary $[\alpha]$ belongs to $H^q(X, \Omega^p)$, suppose we can extend $[\alpha]$ to order $n-1$ in $H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}})$. Denote such element by $[\alpha_{n-1}]$. In the following,

we try to find out the obstruction of the extension of $[\alpha_{n-1}]$ to nth order. Denote $\pi^*(m_0)$ by $\mathcal{M}_0$. Consider the exact sequence

$$0 \to \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \Omega^p_{X_0/B_0} \to \Omega^p_{X_n/B_n} \to \Omega^p_{X_{n-1}/B_{n-1}} \to 0$$

which induces a long exact sequence

$$0 \to H^0(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \Omega^p_{X_0/B_0}) \to H^0(X_n, \Omega^p_{X_n/B_n}) \to H^0(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}}) \\ \to H^1(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \Omega^p_{X_0/B_0}) \to ....$$

The obstruction for $[\alpha_{n-1}]$ comes from the non trivial image of the connecting homomorphism $\delta^* : H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}}) \to H^{q+1}(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \Omega^p_{X_0/B_0})$. We will calculate it by *Čech* calculation.

Cover X by open sets U_i such that, for arbitrary i, U_i is small enough. More precisely, U_i is stein and the following exact sequence splits

$$0 \to \pi_n^*(\Omega_{B_n})(U_i) \to \Omega_{X_n}(U_i) \to \Omega_{X_n/B_n}(U_i) \to 0.$$

So we have a map $\varphi_i : \Omega_{X_n/B_n}(U_i) \to \Omega_{X_n}(U_i)$, such that, $\varphi_i(\Omega_{X_n/B_n}(U_i)) \oplus \pi_n^*(\Omega_{B_n})(U_i) \cong \Omega_{X_n}(U_i)$. Denote by ι_i, ι_i^{-1} the inclusion from $\pi_n^*(\Omega_{B_n})(U_i)$ to $\Omega_{X_n}(U_i)$ and its inverse. Define $d^i_{X_n/B_n}$ by $\varphi_i \circ d_{X_n/B_n} \circ \varphi_i^{-1}$ and $d^i_{B_n}$ by $\iota_i \circ d_{B_n} \circ \iota_i^{-1}$. Then it determines a local decomposition of the exterior differentiation d_{X_n} in $\Omega^\bullet_{X_n}$

$$d_{X_n} = d^i_{B_n} + d^i_{X_n/B_n}.$$

Denote the set of alternating q-cochains β with values in $\mathcal{F}$ by $\mathcal{C}^q(\mathbf{U}, \mathcal{F})$, i.e. to each $q+1$-tuple, $i_0 < i_1 ... < i_q$, β assigns a section $\beta(i_0, i_1, ..., i_q)$ of $\mathcal{F}$ over $U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_q}$.

Let us still using φ_i denote the following map,

$$\begin{aligned} \varphi_i : \pi_n^*(\Omega^r_{B_n}) \wedge \Omega^p_{X_n/B_n}(U_i) \quad &\to \quad \Omega^{p+r}_{X_n}(U_i) \\ \varphi_i(\omega_{i_1} \wedge ... \wedge \omega_{i_r} \wedge \beta_{j_1} \wedge ... \wedge \beta_{j_p}) \quad &= \quad \omega_{i_1} \wedge ... \wedge \omega_{i_r} \wedge \varphi_i(\beta_{j_1}) \wedge ... \wedge \varphi_i(\beta_{j_p}). \end{aligned}$$

Define $\varphi : \mathcal{C}^q(\mathbf{U}, \pi_n^*(\Omega^r_{B_n}) \wedge \Omega^p_{X_n/B_n}) \to \mathcal{C}^q(\mathbf{U}, \Omega^{p+r}_{X_n})$ by

$$\varphi(\beta)(i_0, i_1, ..., i_q) = \varphi_{i_0}(\beta(i_0, i_1, ..., i_q)) \qquad \forall \beta \in \mathcal{C}^q(\mathbf{U}, \pi_n^*(\Omega^r_{B_n}) \wedge \Omega^p_{X_n/B_n}),$$

where $i_0 < i_1 ... < i_q$.

Define the total Lie derivative with respect to B_n

$$L_{B_n} : \mathcal{C}^q(\mathbf{U}, \Omega^p_{X_n}) \to \mathcal{C}^q(\mathbf{U}, \Omega^{p+1}_{X_n})$$

by

$$L_{B_n}(\beta)(i_0, i_1, ..., i_q) = d^i_{B_n}(\beta(i_0, i_1, ..., i_q)) \qquad \forall \beta \in \mathcal{C}^q(\mathbf{U}, \Omega^p_{X_n}),$$

where $i_0 < i_1 ... < i_q$.

Define, for each U_i the total interior product with respect to B_n, I^i : $\Omega^p_{X_n}(U_i) \to \Omega^p_{X_n}(U_i)$ by

$$I^i(\mu dg_1 \wedge dg_2 \wedge ... \wedge dg_p) = \mu \sum_{j=1}^{p} dg_1 \wedge ... \wedge dg_{j-1} \wedge d^i_{B_n}(g_j) \wedge dg_{j+1} \wedge ... \wedge dg_p.$$

When $p = 0$, we put $I^i = 0$.

Define $\lambda : \mathcal{C}^q(\mathbf{U}, \Omega^p_{X_n}) \to \mathcal{C}^{q+1}(\mathbf{U}, \Omega^p_{X_n})$ by

$$(\lambda\beta)(i_0, ..., i_{q+1}) = (I^{i_0} - I^{i_1})\beta(i_1, ..., i_{q+1}) \qquad \forall \beta \in \mathcal{C}^q(\mathbf{U}, \Omega^p_{X_n}).$$

Lemma 3.0

$$\lambda \circ \varphi \equiv \delta \circ \varphi - \varphi \circ \delta$$

mod. $\pi^*_n(\Omega^2_{B_n}) \wedge \Omega^{p-1}_{X_n}$.

Proof. Define $J : C^q(\mathbf{U}, \Omega^p_{X_n/B_n}) \to C^q((U), \Omega^p_{X_n})$ by

$$(J(\beta))(i_0, ..., i_{q+1}) = (-1)(\varphi_{i_0} - \varphi_{i_1})(\beta(i_1, ..., i_{q+1}),$$

where $i_0 < i_1 < ... < i_{q+1}$. For arbitrary β belongs to $C^q(\mathbf{U}, \Omega^p_{X_n/B_n})$,

$$\begin{aligned}(\delta \circ \varphi(\beta))(i_0, ..., i_{q+1}) &= \sum_{j=0}^{q+1}(-1)^j \varphi(\beta)(i_0, ..., \widehat{i_j}, ..., i_{q+1}) \\ &= \varphi_{i_1}(\beta)(i_1, ..., i_{q+1}) \\ &\quad + \sum_{j=1}^{q+1}(-1)^j \varphi_{i_0}(\beta)(i_0, ..., \widehat{i_j}, ..., i_{q+1}),\end{aligned}$$

while

$$
\begin{aligned}
(\varphi \circ \delta(\beta))(i_0, ..., i_{q+1}) &= \varphi(\sum_{j=0}^{q+1}(-1)^j(\beta)(i_0, ..., \widehat{i_j}, ..., i_{q+1})) \\
&= \sum_{j=0}^{q+1}(-1)^j \varphi_{i_0}(\beta)(i_0, ..., \widehat{i_j}, ..., i_{q+1}).
\end{aligned}
$$

So we have $\delta \circ \varphi - \varphi \circ \delta = J$.

Fix $(i_0, ..., i_{q+1})$ and let $\omega = \beta(i_1, ..., i_{q+1})$. We must show that $(I^{i_0} - I_{i_1})(\varphi_{i_1}(\omega)) = (-1)(\varphi_{i_0} - \varphi_{i_1})(w)$ mod $\pi_n^*(\Omega^2_{B_n}) \wedge \Omega^{p-1}_{X_n}$. By linearity, we may suppose $\varphi_{i_1}(\omega) = \mu dg_1 \wedge ... \wedge dg_p$. Then

$$
\begin{aligned}
\varphi_{i_0} &= \mu d^{i^0}_{X_n/B_n}(g_1) \wedge ... \wedge d^{i_0}_{X_n/B_n}(g_p) \\
&= \mu(dg_1 - d^{i^0}_{X_n/B_n}(g_1)) \wedge ... \wedge (dg_p - d^{i_0}_{X_n/B_n}(g_p)) \\
&= \mu dg_1 \wedge ... \wedge dg_p - \sum_{j=1}^{p} \mu dg_1 \wedge ... dg_{j-1} \wedge d^{i_0}_{B_n}(g_j \wedge dg_{j+1} \wedge ... \wedge dg_p \\
&\qquad + \text{terms in } \pi_n^*(\Omega^2_{B_n}) \wedge \Omega^{p-1}_{X_n}.
\end{aligned}
$$

Thus $\varphi_{i_0} \equiv \varphi_{i_1}(\omega) - I^{i_0} \circ \varphi_{i_1}(\omega)$ mod. $\pi_n^*(\Omega^2_{B_n}) \wedge \Omega^{p-1}_{X_n}$, and $I^{i_1} \circ \varphi_{i_1} = 0$. which means $\lambda \circ \varphi \equiv J$ mod. $\pi_n^*(\Omega^2_{B_n}) \wedge \Omega^{p-1}_{X_n}$. □

Now we are ready to calculate the formula for the obstructions. Let $\tilde{\alpha}$ be an element of $\mathcal{C}^q(\mathbf{U}, \Omega^p_{X_n/B_n})$ such that its quotient image in $\mathcal{C}^q(\mathbf{U}, \Omega^p_{X_{n-1}/B_{n-1}})$ is α_{n-1}. Then $\delta^*([\alpha_{n-1}])$= $[\delta(\tilde{\alpha})]$ which is an element of $H^{q+1}(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \Omega^p_{X_0/B_0}) \cong \mathrm{m}_0^n/\mathrm{m}_0^{n+1} \otimes H^{q+1}(X, \Omega^p_{X_0/B_0})$.

Denote r_{X_n} the restriction to the complex space X_n. In order to give the obstructions an explicit calculation, we need to consider the following map $\rho : H^q(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \Omega^p_{X_0/B_0}) \to H^q(X_{n-1}, \pi^*_{n-1}(\Omega_{B_n|B_{n-1}}) \wedge \Omega^p_{X_{n-1}/B_{n-1}})$. which is defined by $\rho[\sigma] = [\varphi^{-1} \circ r_{X_{n-1}} \circ L_{B_n} \circ \varphi(\sigma)]$.

Lemma 3.1 *The map:* $\rho : H^q(X, \mathcal{M}_0^n/\mathcal{M}_0^{n+1} \otimes \Omega^p_{X_0/B_0}) \to$ $H^q(X_{n-1}, \pi^*_{n-1}(\Omega_{B_n|B_{n-1}}) \wedge \Omega^p_{X_{n-1}/B_{n-1}})$ *is well defined.*

Proof. At first, we need to show that if σ is closed, then $\varphi^{-1} \circ r_{X_{n-1}} \circ L_{B_n} \circ \varphi(\sigma)$ is closed, which is equivalent to show that $\delta \circ r_{X_{n-1}} \circ L_{B_n} \circ \varphi(\sigma) \equiv 0$ mod. $\pi^*_{n-1}(\Omega^2_{B_n|B_{n-1}}) \wedge \Omega^{p-1}_{X_n|X_{n-1}}$.

Note that $d_{X_n} \circ \delta = -\delta \circ d_{X_n}$. Then

$$\begin{aligned} \delta \circ r_{X_{n-1}} \circ L_{B_n} \circ \varphi(\sigma) &= r_{X_{n-1}} \circ \delta \circ L_{B_n} \circ \varphi(\sigma) \\ &= -r_{X_{n-1}} \circ (\delta \circ d^{\cdot}_{X_n/B_n} + d^{\cdot}_{X_n/B_n} \circ \delta + L_{B_n} \circ \delta) \circ \varphi(\sigma). \end{aligned}$$

Since

$$L_{B_n} \circ \delta \circ \varphi(\sigma) \equiv L_{B_n} \circ (\delta \circ \varphi - \lambda \circ \varphi)(\sigma) \equiv L_{B_n} \circ \varphi \circ \delta(\sigma) = 0$$

and

$$r_{X_{n-1}} \circ (\delta \circ d^{\cdot}_{X_n/B_n} + d^{\cdot}_{X_n/B_n} \circ \delta) \circ \varphi(\sigma) = 0,$$

we have $\delta \circ r_{X_{n-1}} \circ L_{B_n} \circ \varphi(\sigma) \equiv 0 \text{ mod. } \pi^*_{n-1}(\Omega^2_{B_n|B_{n-1}}) \wedge \Omega^{p-1}_{X_n|X_{n-1}}$.

Next we need to show that if σ is belongs to $C^q(\mathbf{U}, \mathcal{M}^n_0/\mathcal{M}^{n+1}_0 \otimes \Omega^p_{X_0/B_0})$, then $\varphi^{-1} \circ r_{X_{n-1}} \circ L_{B_n} \circ \varphi \circ \delta(\sigma)$ is exact. In fact, as the calculation above: $r_{X_{n-1}} \circ L_{B_n} \circ \varphi \circ \delta(\sigma) \equiv -r_{X_{n-1}} \circ (\delta \circ d^{\cdot}_{X_n/B_n} + d^{\cdot}_{X_n/B_n} \circ \delta + \delta \circ L_{B_n}) \circ \varphi(\sigma) = -\delta \circ r_{X_{n-1}} \circ L_{B_n} \circ \varphi(\sigma)$.

□

In general, the map ρ is not injective. However, as we mentioned at the end of the previous section. The "real" obstructions are $o^q_{n,n-1}([\alpha_{n-1}])$, but not $o^q_n([\alpha_{n-1}])$. So we don't need ρ to be injective. In the following, we will explain that $\rho([\delta(\tilde{\alpha})])$ is exactly the "real" obstructions we need. In fact,

$$H^q(X_{n-1}, \pi^*_{n-1}(\Omega_{B_n|B_{n-1}}) \wedge \Omega^p_{X_{n-1}/B_{n-1}}) = (\Omega_{B_n|B_{n-1}}) \otimes_{\mathcal{O}_{B_{n-1}}} H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}}).$$

Let $m = dim_{\mathbb{C}} B$, let $t_i, i = 0...m$ be the local coordinates of B. Then $\rho([\delta(\tilde{\alpha})])$ can be written as: $\sum_{i=0}^m dt_i \otimes \tilde{\alpha}_i$, where $\tilde{\alpha}_i \in H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}})$. For a certain direction $\frac{\partial}{\partial t_i}$, suppose $\tilde{\alpha}_i \neq 0$. Then by a simple calculation, it is not difficult to check that $\tilde{\alpha}_i = constant[\delta(\tilde{\alpha})/t_i]$ in $H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}})$. While $[\delta(\tilde{\alpha})/t_i]$ is exactly the obstruction $o^q_{n,n-1}([\alpha_{n-1}])$ in the direction of $\frac{\partial}{\partial t_i}$ we mentioned in the previous section.

Now consider the following exact sequence. The connecting homomorphism of the associated long exact sequence gives the Kodaira-Spencer class

of order n [4 1.3.2],

$$0 \to \pi^*_{n-1}(\Omega_{B_n|B_{n-1}}) \to \Omega_{X_n|X_{n-1}} \to \Omega_{X_{n-1}/B_{n-1}} \to 0.$$

By wedge the above exact sequence with $\Omega^{p-1}_{X_{n-1}/B_{n-1}}$, we get a new exact sequence. The connecting homomorphism of such exact sequence gives us a map from $H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}})$ to $H^{q+1}(X_{n-1}, \pi^*(\Omega_{B_n|B_{n-1}}) \wedge \Omega^{p-1}_{X_{n-1}/B_{n-1}})$. Denote such map by $\kappa_n \llcorner$, for such map is simply the inner product with the Kodaira-Spencer class of order n. By the definition and simply calculation it is not difficult to proof the following lemma.

Lemma 3.2 *Let θ be an element of $H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}})$, let $\tilde{\theta}$ be an element of $\mathcal{C}^q(\mathbf{U}, \Omega^p_{X_n/B_n})$ such that its quotient image is θ. Then $[\kappa_n \llcorner \theta]$ is equal to $[\varphi^{-1} \circ r_{X_{n-1}} \circ \delta \circ \varphi(\tilde{\theta})]$.*

Let us come back to the problem we discussed, we have

$$\begin{aligned}
r_{X_{n-1}} \circ L_{B_n} \circ \varphi \circ \delta(\tilde{\alpha}) &\equiv r_{X_{n-1}} \circ L_{B_n} \circ (\delta \circ \varphi - \lambda \circ \varphi)(\tilde{\alpha}) \\
&\equiv r_{X_{n-1}} \circ L_{B_n} \circ \delta \circ \varphi(\tilde{\alpha}) \\
&\equiv -r_{X_{n-1}} \circ (d^{\cdot}_{X_n/B_n} \circ \delta + \delta \circ d^{\cdot}_{X_n/B_n} + \delta \circ L_{B_n}) \circ \varphi(\tilde{\alpha}) \\
&\equiv -r_{X_{n-1}} \circ (d^{\cdot}_{X_n/B_n} \circ \delta + \delta \circ d^{\cdot}_{X_n/B_n}) \circ \varphi(\tilde{\alpha}) \\
&\quad -\delta \circ r_{X_{n-1}} \circ L_{B_n}) \circ \varphi(\tilde{\alpha}).
\end{aligned}$$

Therefore

$$\begin{aligned}
[r_{X_{n-1}} \circ L_{B_n} \circ \varphi \circ \delta(\tilde{\alpha})] &= [-r_{X_{n-1}} \circ (d^{\cdot}_{X_n/B_n} \circ \delta + \delta \circ d^{\cdot}_{X_n/B_n}) \circ \varphi(\tilde{\alpha})] \\
&= -[d^{\cdot}_{X_{n-1}/B_{n-1}} \circ r_{X_{n-1}} \delta \circ \varphi(\tilde{\alpha}) + r_{X_{n-1}} \circ \delta \circ d^{\cdot}_{X_n/B_n} \circ \varphi(\tilde{\alpha})] \\
&= -[d^{\cdot}_{X_{n-1}/B_{n-1}} \circ \varphi \circ \varphi^{-1} \circ r_{X_{n-1}} \delta \circ \varphi(\tilde{\alpha}) \\
&\quad +r_{X_{n-1}} \circ \delta \circ \varphi \circ d_{X_n/B_n}(\tilde{\alpha})] \\
&= -[\varphi \circ d_{X_{n-1}/B_{n-1}} \circ \varphi^{-1} \circ r_{X_{n-1}} \delta \circ \varphi(\tilde{\alpha}) \\
&\quad +r_{X_{n-1}} \circ \delta \circ \varphi \circ (d_{X_{n-1}/B_{n-1}}(\alpha_{n-1}))] \\
&= -[d_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \alpha_{n-1} + \kappa_n \llcorner \circ d_{X_{n-1}/B_{n-1}}(\alpha_{n-1})].
\end{aligned}$$

From the discussion above, we get the main theorem of this paper.

Theorem 3.3 *Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the nth order deformation of X. For arbitrary $[\alpha]$ belongs to $H^q(X, \Omega^p)$, suppose we can extend $[\alpha]$ to order $n-1$ in $H^q(X_{n-1}, \Omega^p_{X_{n-1}/B_{n-1}})$. Denote such element by $[\alpha_{n-1}]$. The obstruction of the extension of $[\alpha]$ to nth order is given by:*

$$o_{n,n-1}(\alpha_{n-1}) = d_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner (\alpha_{n-1}) + \kappa_n \llcorner \circ d_{X_{n-1}/B_{n-1}}(\alpha_{n-1}),$$

where κ_n is the nth order Kodaira-Spencer class and $d_{X_{n-1}/B_{n-1}}$ is the relative differential operator of the $n-1$th order deformation.

From the theorem, we can get the following corollary immediately.

Corollary 3.4 *Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Suppose that up to order n, the d_1 of the Frölicher spectral sequence vanishes. For arbitrary $[\alpha]$ belongs to $H^q(X, \Omega^p)$, it can be extended to order $n+1$ in $H^q(X_{n+1}, \Omega^p_{X_{n+1}/B_{n+1}})$.*

4 An Example

In this section, we will use the formula in previous section to study the jumping of the Hodge numbers $h^{p,q}$ of small deformations of Iwasawa manifold. It was Kodaira who first calculated small deformations of Iwasawa manifold [2]. In the first part of this section, let us recall his result.

Set

$$G = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} ; z_i \in \mathbb{C} \right\} \cong \mathbb{C}^3$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} ; \omega_i \in \mathbb{Z} + \mathbb{Z}\sqrt{-1} \right\}.$$

The multiplication is defined by

$$\begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2 + \omega_2 & z_3 + \omega_2 z_1 + \omega_3 \\ 0 & 1 & z_1 + \omega_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$X = G/\Gamma$ is called Iwasawa manifold. We may consider $X = \mathbb{C}^3/\Gamma$. $g \in \Gamma$

operates on $\mathbb{C}^3$ as follows:

$$z_1' = z_1 + \omega_1, \qquad z_2' = z_2 + \omega_2, \qquad z_3' = z_3 + \omega_1 z_2 + \omega_3$$

where $g = (\omega_1, \omega_2, \omega_3)$ and $z' = z \cdot g$. There exist holomorphic 1-froms $\varphi_1, \varphi_2, \varphi_3$ which are linearly independent at every point on X and are given by

$$\varphi_1 = dz_1, \qquad \varphi_2 = dz_2, \qquad \varphi_3 = dz_3 - z_1 dz_2,$$

so that

$$d\varphi_1 = d\varphi_2 = 0, \qquad d\varphi_3 = -\varphi_1 \wedge \varphi_2.$$

On the other hand we have holomorphic vector fields $\theta_1, \theta_2, \theta_3$ on X given by

$$\theta_1 = \frac{\partial}{\partial z_1}, \qquad \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \qquad \theta_3 = \frac{\partial}{\partial z_3},$$

It is easily seen that

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \qquad [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0.$$

in view of Theorem 3 in [2], $H^1(X, \mathcal{O})$ is spanned by $\overline{\varphi}_1, \overline{\varphi}_2$. Since Θ is isomorphic to $\mathcal{O}^3$, $H^1(X, TX)$ is spanned by $\theta_i \overline{\varphi}_\lambda, i = 1, 2, 3, \lambda = 1, 2$.

The small deformation o f X is given by

$$\psi(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \overline{\varphi}_\lambda t - (t_{11} t_{22} - t_{21} t_{12}) \theta_3 \overline{\varphi}_3 t^2.$$

We summarize the numerical characters of deformations. The deformations are divided into the following three classes:

i) $t_{11} = t_{12} = t_{21} = t_{22} = 0$, X_t is a parallelisable manifold.
ii) $t_{11} t_{22} - t_{21} t_{12} = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0,0,0,0)$, X_t is not parallelisable.
iii) $t_{11} t_{22} - t_{21} t_{12} \neq 0$, X_t is not parallelisable.

	$h^{1,0}$	$h^{0,1}$	$h^{2,0}$	$h^{1,1}$	$h^{0,2}$	$h^{3,0}$	$h^{2,1}$	$h^{1,2}$	$h^{3,0}$
i)	3	2	3	6	2	1	6	6	1
ii)	2	2	2	5	2	1	5	5	1
iii)	2	2	1	5	2	1	4	4	1

Now let us explain the jumping phenomenon of the Hodge number by using the obstruction formula. From Corollary 4.3 in [6], it follows that the Dolbeault cohomology groups are:

$$\begin{aligned}
H^0(X,\Omega) &= Span\{[\varphi_1],[\varphi_2],[\varphi_3]\},\\
H^1(X,\mathcal{O}) &= Span\{[\overline{\varphi}_1],[\overline{\varphi}_2]\},\\
H^0(X,\Omega^2) &= Span\{[\varphi_1\wedge\varphi_2],[\varphi_2\wedge\varphi_3],[\varphi_3\wedge\varphi_1]\},\\
H^1(X,\Omega) &= Span\{[\varphi_i\wedge\overline{\varphi}_\lambda]\}, i=1,2,3,\lambda=1,2,\\
H^2(X,\mathcal{O}) &= Span\{[\overline{\varphi}_2\wedge\overline{\varphi}_3],[\overline{\varphi}_3\wedge\overline{\varphi}_1]\},\\
H^0(X,\Omega^3) &= Span\{[\varphi_1\wedge\varphi_2\wedge\varphi_3]\},\\
H^1(X,\Omega^2) &= Span\{[\varphi_i\wedge\varphi_j\wedge\overline{\varphi}_\lambda]\}, i,j=1,2,3, i<j, \lambda=1,2,\\
H^2(X,\Omega^1) &= Span\{[\varphi_i\wedge\overline{\varphi}_2\wedge\overline{\varphi}_3],[\varphi_j\wedge\overline{\varphi}_1\wedge\overline{\varphi}_3]\}, i,j=1,2,3,\\
H^3(X,\mathcal{O}) &= Span\{[\overline{\varphi}_1\wedge\overline{\varphi}_2\wedge\overline{\varphi}_3]\},
\end{aligned}$$

For example, let us first consider $h^{2,0}$, in the ii) class of deformation. The Kodaira-Spencer class of the this deformation is $\psi_1(t)=\sum_{i=1}^3\sum_{\lambda=1}^2 t_{i\lambda}\theta_i\overline{\varphi}_\lambda$, with $t_{11}t_{22}-t_{21}t_{12}=0$. It is easy to check that $o_1(\varphi_1\wedge\varphi_2)=\partial(int(\psi_1(t))(\varphi_1\wedge\varphi_2)-int(\psi_1(t))(\partial(\varphi_1\wedge\varphi_2))=0$, $o_1(t_{11}\varphi_2\wedge\varphi_3-t_{21}\varphi_1\wedge\varphi_3)=\partial((t_{11}t_{22}-t_{21}t_{12})\varphi_3\wedge\overline{\varphi}_2)=0$, and $o_1(\varphi_2\wedge\varphi_3)=-t_{21}\varphi_1\wedge\varphi_2\wedge\overline{\varphi}_1-t_{22}\varphi_1\wedge\varphi_2\wedge\overline{\varphi}_2$, $o_1(\varphi_1\wedge\varphi_3)=-t_{11}\varphi_1\wedge\varphi_2\wedge\overline{\varphi}_1-t_{21}\varphi_1\wedge\varphi_2\wedge\overline{\varphi}_2$. Therefore, we have shown that for an element of the subspace $Span\{[\varphi_1\wedge\varphi_2],[t_{11}\varphi_2\wedge\varphi_3-t_{21}\varphi_1\wedge\varphi_3]\}$, the first order obstruction is trivial, while, since $(t_{11},t_{12},t_{21},t_{22})\neq(0,0,0,0)$, at least one of the obstruction $o_1(\varphi_2\wedge\varphi_3)$, $o_1(\varphi_1\wedge\varphi_3)$ is non trivial which partly explain why the Hodge number $h^{2,0}$ jumps from 3 to 2. For another example, let us consider $h^{1,2}$, in the ii) class of deformation. It is easy to check that for an element of the subspace (the dimension of such a subspace is 5) $Span\{[\varphi_i\wedge\overline{\varphi}_\lambda\wedge\overline{\varphi}_3],[t_{12}\varphi_3\wedge\overline{\varphi}_2\wedge\overline{\varphi}_3-t_{11}\varphi_3\wedge\overline{\varphi}_1\wedge\overline{\varphi}_3]\}, i=1,2,\lambda=1,2$, the first order obstruction is trivial, while at least one of the obstruction $o_1(\varphi_3\wedge\overline{\varphi}_2\wedge\overline{\varphi}_3)$, $o_1(\varphi_3\wedge\overline{\varphi}_1\wedge\overline{\varphi}_3)$ is non trivial.

Remark 1 It is easy to see that, in the ii) or iii) class of deformation, the first order obstruction for any element in $H^1(X,\Omega)$ is trivial. The reason of Hodge number $h^{1,1}$'s jumping from 6 to 5 comes from the existence of the second class obstructed elements $o_1(\varphi_3)$. After simple calculation, it is not difficult to get the structure equation of $X_t, t\neq 0$.

$$\begin{cases} d\varphi_1 = 0, \\ d\varphi_2 = 0, \\ d\varphi_3 = -\varphi_1 \wedge \varphi_2 + to_1(\varphi_3), \qquad i = 1, 2, \lambda = 1, 2, \end{cases}$$

which can be considered an example of proposition 2.5.

Remark 2 From the example we discussed above, it is not difficult to find out the following fact. Let X be an non-Kähler nilpotent complex parallelisable manifold whose dimension is more than 2, and $\phi : \mathcal{X} \to B$ be the versal deformation family of X. Then the Hodge number $h^{1,0}$ will jump in a neighborhood of $0 \in B$. In fact, let $\varphi_i, i = 1...n, n = dim_{\mathbb{C}}(X)$ be the linearly independent holomorphic 1-forms of X. By the theorem 3 of [2], $H^1(X, \mathcal{O})$ is spanned by a subset of $\{\overline{\varphi}_i\}, i = 1..n$. So we have $\partial : H^1(X, \mathcal{O}) \to H^1(X, \Omega)$ is trivial, which means one term of the first order obstruction of the holomorphic 1-forms vanishes. Let $\theta_i, i = 1...n$ be the dual of φ_i, which are linearly independent holomorphic vector fields. Since X is non-Kähler, which means X is not a torus, there exists φ_i such that $\partial\varphi_i \neq 0$. Since X is nilpotent, there exist φ_j such that $\partial\varphi_j = 0$. Assume that $\partial\varphi_i = A\varphi_k \wedge \varphi_l + ...$ with $A \neq 0$. Consider $\theta_k\overline{\varphi}_j$ in $H^1(X, TX)$. It is easy to check that $o_1(\partial\varphi_i, \theta_k\overline{\varphi}_j) \neq 0$.

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